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Research article

Soliton solutions and stability analysis of the stochastic nonlinear reaction-diffusion equation with multiplicative white noise in soliton dynamics and optical physics

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Abstract: In this article, we explored the stochastic nonlinear reaction-diffusion (RD) equation under the influence of multiplicative white noise. To obtain novel soliton solutions, we employed two powerful analytical techniques: the unified Riccati equation expansion method and the modified Kudryashov method. These methods yield a diverse set of soliton solutions, including combo-dark solitons, dark solitons, singular solitons, combo-bright-singular solitons, and periodic wave solutions. We also performed a comprehensive stability analysis of the stochastic nonlinear RD equation with multiplicative white noise. The findings provide valuable insights into the behavior of solitons in stochastic nonlinear systems, with significant implications for fields such as mathematical physics, nonlinear science, and applied mathematics. These results hold particular relevance for soliton dynamics in optical physics, where they can be applied to improve understanding of wave propagation in noisy environments, signal transmission, and the design of robust optical communication systems.

Keywords: solitons; stochastic; nonlinear reaction-diffusion; multiplicative white noise; soliton dynamics; optical physics; wave propagation
Mathematics Subject Classification: 34L30, 35A24, 35B35, 35C08, 74J35

1. Introduction

Research into soliton solutions of nonlinear partial differential equations (NLPDEs) has attracted significant attention due to their wide-ranging applications in fields such as fluid dynamics, plasma physics, and biological systems. Solitons are stable, localized wave solutions that retain their shape during propagation and interaction. The study of solitons in time-fractional systems extends traditional soliton theory into fractional calculus, offering a richer framework to describe wave phenomena in complex systems [1]. For example, the behavior of solitons in the stochastic Chaffee-Infante equation, a model for reaction-diffusion systems, has been explored in [2]. Recent research has applied various computational techniques to solve NLPDEs and uncover different types of soliton solutions. Alraddadi et al. (2024) used an efficient expansion method to find new soliton solutions for two nonlinear PDEs [3]. Mhadhbi et al. (2024) combined classical methods, such as the inverse scattering transform, with innovative approaches to derive exact solutions for nonlinear PDEs [4]. Other studies have incorporated methods like the bilinear neural network technique, as shown in the work by Zhang et al. (2024) and Ye et al. (2024), which focus on symbolic computation for solving NLPDEs, including time-fractional equations [5, 6]. Moreover, the use of physics-informed neural networks (PINN) has expanded the approach to solving complex PDEs, as demonstrated in works by Hu et al. (2024) and Linghu et al. (2025), which applied PINN methods to various solid mechanics and composite material problems [7, 8]. Sarker et al. (2024) explored soliton solutions to nonlinear wave equations using modern methods such as the sine-cosine and exp-function methods [9], contributing to the broad spectrum of soliton solutions, including bright, dark, singular, and periodic solitons. These solitons are pivotal in several areas such as optics, plasma physics, fluid dynamics, nonlinear optics, and quantum mechanics [10, 11], where they model phenomena like light propagation, shock waves, rogue waves [12], and shallow water dynamics [13].

The reaction-diffusion equation (RDE) has garnered substantial attention in recent years due to its widespread relevance across various scientific fields, including physics, chemistry, and biology. This interest is primarily fueled by the fascinating characteristics and diverse properties of the solutions it provides [14]. The evolution of dynamic quantities in these systems is described by nonlinear partial differential equations (NLPDEs). Diffusion and reaction processes play a crucial role in the behavior of many systems, including those in plasma and semiconductor physics. The simultaneous presence of both processes often leads to solutions that are only valid when they both influence the system's dynamics [15, 16]. Over time, the nonlinear RDE has been refined to capture a wider range of complex behaviors. Key advancements include the incorporation of nonlinearities into the reaction terms, which can give rise to intricate phenomena such as pattern formation, traveling waves, and chaotic behavior [17]. To study these systems, researchers have developed a variety of analytical and numerical methods, yielding valuable insights and contributing significantly to fields ranging from ecology to materials science [18–20].

Numerous studies have explored the impact of multiplicative noise on soliton solutions in various nonlinear equations. Key works include those by Abdelrahman et al. (2021) on the nonlinear Schrödinger equation [21], Albosaily et al. (2020) on the stochastic chiral nonlinear Schrödinger equation [22], and Mohammed et al. (2022) on the Hirota-Maccari system [23]. Further studies, such as Mohammed et al. (2021) on the Ginzburg-Landau equation [24], and the coupled Konno-Oono and Nizhnik-Novikov-Veselov systems [25, 26], show how noise alters soliton behavior. Additionally, research on the stochastic Burgers' equation [27], the Konno-Oono system in a magnetic field [28], and the time-fractional Gray-Scott model [29] highlights the significant role of noise in modifying soliton structures across various systems. These studies emphasize the robustness of soliton solutions in noisy environments, demonstrating their importance in understanding complex systems governed by stochastic dynamics.

The stochastic nonlinear RD equation with multiplicative white noise represents a further advancement of this concept [30]. By integrating stochastic processes, specifically multiplicative white noise, into the traditional nonlinear reaction-diffusion framework, this extension enables the modeling of random fluctuations impacting the system. This enhancement offers a more thorough representation of phenomena affected by inherent uncertainties.

In this work, we present a novel approach to solving the stochastic nonlinear reaction-diffusion equation with multiplicative white noise. By employing both the unified Riccati equation expansion method [31] and the modified Kudryashov method [32], we derive a diverse set of soliton solutions, including combo-dark solitons, dark solitons, singular solitons, combo-bright-singular solitons, and periodic wave solutions. This dual-method approach enhances the scope of soliton analysis, providing insights beyond existing studies. Unlike prior studies that may not fully address soliton stability under noise, our stability analysis demonstrates that certain solitons maintain structural integrity even in stochastic settings. This highlights the robustness of specific soliton solutions, which is a critical finding for real-world applications where noise is inevitable.

1.1. Principal model

In this article, we investigate the stochastic nonlinear RD equation with multiplicative white noise, which is formulated as follows [30]:

$$q_t = \left(aq^{n-1}q_x\right)_x - bq + cq^n + \sigma q\frac{dW(t)}{dt}. \tag{1}$$

In this context, $q(x, t)$ denotes the population density at a specific position x and time t. The constants a, b, and c each uniquely influence the system's dynamics. Specifically, $a \neq 0$ ensures that the reaction term remains significant, while n (with $n \neq 1$) characterizes the nonlinearity of the equation. Additionally, σ represents the noise strength coefficient, and $W(t)$ denotes the standard Wiener process. The term "white noise" is mathematically defined as $\frac{dW(t)}{dt}$. The stochastic process is characterized by the following properties:

(i) The function $W(t)$ is continuous for $t \geq 0$.

(ii) The difference $W(t) - W(s)$, for $t > s$, is normally distributed with a mean of zero and a variance of $t - s$.

This process is commonly referred to as Brownian motion.

The structure of this paper is outlined as follows: In Section 2, we present the mathematical analysis. Section 3 addresses the solution of Eq (1) using the unified Riccati equation expansion method. In Section 4, we apply the modified Kudryashov method. Section 5 focuses on the stability analysis of Eq (1). The results and discussion are introduced in Section 6, and the conclusions are provided in Section 7.

2. Mathematical analysis

To achieve this objective, we assume the following about the formal solution to Eq (1):

$$q(x,t) = \Phi(z)\exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{2}$$

and

$$z = x - v\,t, \tag{3}$$

where $\Phi(z)$ represents a traveling wave solution moving with velocity v. This function describes the shape of the wave as it propagates through space. σ is a constant representing the strength of the noise in the system. It quantifies the influence of stochastic effects on the solution. $W(t)$ denotes the standard Wiener process (or Brownian motion), which models the random fluctuations over time due to noise. The term $\sigma W(t)$ represents the contribution of these stochastic fluctuations to the solution. The exponential factor $\exp\left[\sigma W(t) - \frac{(n-1)}{2}\sigma^2 t\right]$ adjusts the amplitude of the solution, accounting for the combined effects of noise and the nonlinearity characterized by the integer n. t is time, and x is the spatial coordinate. This form of the solution suggests that the population density evolves as a traveling wave, with its amplitude modulated by both stochastic effects and nonlinearity.

By substituting Eqs (2) and (3) into Eq (1), we obtain the following result:

$$v\Phi'(z)+\left(\frac{n-1}{2}\sigma^2-b\right)\Phi(z)+\left[a(n-1)\Phi^{n-2}(z)\Phi'^2(z)+a\Phi^{n-1}(z)\Phi''(z)+c\Phi^n(z)\right]e^{-\frac{(n-1)^2}{2}\sigma^2 t}e^{(n-1)\sigma W(t)}=0. \tag{4}$$

Taking the expectation on both sides of Eq (4), where $\Phi(z)$ is a deterministic function, yields:

$$v\Phi'(z)+\left(\frac{n-1}{2}\sigma^2-b\right)\Phi(z)+\left[a\,(n-1)\Phi^{n-2}(z)\Phi'^2(z)+a\Phi^{n-1}(z)\Phi''(z)+c\Phi^n(z)\right]e^{-\frac{(n-1)^2}{2}\sigma^2 t}\mathbb{E}\left(e^{(n-1)\sigma W(t)}\right)=0. \tag{5}$$

Recognizing a fundamental property of the expectation operator applied to exponential functions of standard normal random variables, we note that $\mathbb{E}(e^{\gamma Z}) = e^{\frac{\gamma^2}{2}t}$, where γ is any real number and Z is a standard normal random variable. This property remains valid for any γ and standard normal Z. Thus, $\mathbb{E}\left(e^{(n-1)\sigma W(t)}\right) = e^{\frac{(n-1)^2}{2}\sigma^2 t}$. As a results, Eq (5) is represented as:

$$a\Phi(z)\Phi''(z) + a\,(n-1)\,\Phi'^2(z) + v\Phi'(z)\Phi^{2-n}(z) + c\Phi^2(z) + \left(\frac{n-1}{2}\sigma^2 - b\right)\Phi^{3-n}(z) = 0. \tag{6}$$

By balancing $\Phi(z)\Phi''(z)$ and $\Phi'(z)\Phi^{2-n}(z)$ in Eq (6), we derive the balance $N = \frac{1}{1-n}$. Since N is not an integer, we proceed by taking:

$$\Phi(z) = [g(z)]^{\frac{1}{1-n}}. \tag{7}$$

Substituting (7) into Eq (6), we derive a new equation:

$$\begin{aligned} &2a(n-1)g(z)g''(z) + 2a(1-2n)g'^2(z) + 2v(n-1)g^2(z)g'(z) \\ -\;& 2c(n-1)^2 g^2(z) + (n-1)^2\left[2b - (n-1)\sigma^2\right]g^3(z) = 0. \end{aligned} \tag{8}$$

The following approaches will then be used to solve Eq (8).

3. Unified Riccati equation expansion method (UREEM)

The UREEM provides multiple soliton solutions, including combo, dark, singular, and periodic solitons, making it widely applicable. The method is straightforward and systematic, which makes it easy to implement for various nonlinear PDEs. It is versatile, as it captures a range of physical phenomena by producing different types of soliton solutions. Additionally, UREEM offers exact solutions, providing valuable analytical insights into the systems being studied. One major limitation of the UREEM is its inability to deduce bright soliton solutions, which restricts its applicability to certain systems. The method depends on the Riccati equation structure, which may not always align with the requirements of all nonlinear systems. Furthermore, the solutions obtained can involve complex coefficients or parameters, often necessitating additional simplification. According to this method [31], it is assumed that Eq (8) has the following formal solution:

$$g(z) = \sum_{l=0}^{N} \zeta_l F^l(z), \tag{9}$$

where ζ_l are constants and N is a positive integer with the condition $\zeta_N \neq 0$. Consequently, $F(z)$ satisfies the Riccati equation:

$$F'(z) = \sum_{j=0}^{2} \pi_j F^j(z), \tag{10}$$

such that π_j are constants, with the condition $\pi_2 \neq 0$. Equation (10) has the following fractional solutions:

$$F(z) = -\frac{\pi_1}{2\pi_2} - \frac{\sqrt{\pi_1^2-4\pi_0\pi_2}}{2\pi_2}\left[\frac{A_1 \tanh\left(\frac{z}{2}\sqrt{\pi_1^2-4\pi_0\pi_2}\right)+A_2}{A_1+A_2 \tanh\left(\frac{z}{2}\sqrt{\pi_1^2-4\pi_0\pi_2}\right)}\right], \text{ if } \pi_1^2 - 4\pi_0\pi_2 > 0 \text{ and } A_1^2 + A_2^2 \neq 0, \tag{11}$$

$$F(z) = -\frac{\pi_1}{2\pi_2} + \frac{\sqrt{-(\pi_1^2-4\pi_0\pi_2)}}{2\pi_2}\left[\frac{A_3 \tan\left(\frac{z}{2}\sqrt{-(\pi_1^2-4\pi_0\pi_2)}\right)-A_4}{A_3+A_4 \tan\left(\frac{z}{2}\sqrt{-(\pi_1^2-4\pi_0\pi_2)}\right)}\right], \text{ if } \pi_1^2 - 4\pi_0\pi_2 < 0 \text{ and } A_3^2 + A_4^2 \neq 0, \tag{12}$$

$$F(z) = -\frac{\pi_1}{2\pi_2} + \frac{1}{\pi_2 z + A_5}, \text{ if } \pi_1^2 - 4\pi_0\pi_2 = 0, \tag{13}$$

where $A_r (r = 1, 2, ..., 5)$ are constants.

Balancing $g(z)g''(z)$ with $g^2(z)g'(z)$ in Eq (8), one gets $N = 1$. From Eq (9), the solution to Eq (8) takes the following form:

$$g(z) = \zeta_0 + \zeta_1 F(z), \tag{14}$$

where ζ_0 and $\zeta_1$are constants, provided $\zeta_1 \neq 0$. Substituting Eqs (10) and (14) into Eq (8) yields the following algebraic equations:

$$
\left.\begin{array}{l}
2\,\zeta_1^3 n v\pi_2 - 2\,\zeta_1^3 v\pi_2 - 2\,a\zeta_1^2\pi_2^2 = 0,\\[2mm]
2\,b\zeta_1^3 n^2 + 2\,\zeta_1^3\pi_1 n v + 4\,\zeta_1^2 n v\pi_2\zeta_0 - 2\,a\zeta_1^2\pi_1\pi_2 - 4\,b\zeta_1^3 n - \sigma^2\zeta_1^3 n^3 + 3\,\sigma^2\zeta_1^3 n^2 - 3\,\sigma^2\zeta_1^3 n\\
-4\,\zeta_1^2 v\pi_2\zeta_0 - 2\,a\zeta_1^2\pi_1 n\pi_2 + \sigma^2\zeta_1^3 - 4\,a\zeta_1\pi_2^2\zeta_0 - 2\,\zeta_1^3\pi_1 v + 2\,b\zeta_1^3 + 4\,a\zeta_1 n\pi_2^2\zeta_0 = 0,\\[2mm]
-2\,\zeta_1^2 c - 2\,v\zeta_1^3\pi_0 - 2\,v\zeta_1\pi_2\zeta_0^2 - 12\,b\zeta_1^2 n\zeta_0 - 6\,a\zeta_1\pi_1\pi_2\zeta_0 - 2\,\zeta_1^2 c n^2 - 2\,a\zeta_1^2\pi_1^2 n\\
+6\,b\zeta_1^2\zeta_0 - 3\,\sigma^2\zeta_1^2 n^3\zeta_0 + 3\,\sigma^2\zeta_1^2\zeta_0 - 9\,\sigma^2\zeta_1^2 n\zeta_0 + 2\,v\zeta_1^3 n\pi_0 + 4\,\zeta_1^2\pi_1 n v\zeta_0 + 2\,\zeta_1 n v\pi_2\zeta_0^2\\
-4\,a\zeta_1^2 n\pi_0\pi_2 + 9\,\sigma^2\zeta_1^2 n^2\zeta_0 + 6\,a\zeta_1\pi_1 n\pi_2\zeta_0 + 4\,\zeta_1^2 c n - 4\,\zeta_1^2\pi_1 v\zeta_0 + 6\,b\zeta_1^2 n^2\zeta_0 = 0,\\[2mm]
9\,\sigma^2\zeta_1 n^2\zeta_0^2 - 2\,\zeta_1\pi_1 v\zeta_0^2 + 2\,\zeta_1\pi_1 n v\zeta_0^2 - 4\,\zeta_1 c\zeta_0 + 2\,a\zeta_1\pi_1^2 n\zeta_0 + 3\,\sigma^2\zeta_1\zeta_0^2 - 4\,a\zeta_1\zeta_0\pi_0\pi_2\\
-2\,a\zeta_1\pi_1^2\zeta_0 + 8\,\zeta_1 c n\zeta_0 - 4\,\zeta_1 c n^2\zeta_0 - 3\,\sigma^2\zeta_1 n^3\zeta_0^2 - 4\,v\zeta_1^2\zeta_0\pi_0 - 9\,\sigma^2\zeta_1 n\zeta_0^2 + 6\,b\zeta_1\zeta_0^2\\
-12\,b\zeta_1 n\zeta_0^2 + 2\,a\zeta_1^2\pi_0\pi_1 + 6\,b\zeta_1 n^2\zeta_0^2 - 6\,a\zeta_1^2 n\pi_0\pi_1 + 4\,v\zeta_1^2\zeta_0 n\pi_0 + 4\,a\zeta_1\zeta_0 n\pi_0\pi_2 = 0,\\[2mm]
2\,a\zeta_1\zeta_0 n\pi_0\pi_1 - 4\,a\zeta_1^2 n\pi_0^2 - 2\,c n^2\zeta_0^2 + 4\,c\zeta_0^2 n - 2\,v\zeta_1\zeta_0^2\pi_0 - 2\,c\zeta_0^2 + 3\,n^2\sigma^2\zeta_0^3 - 4\,n b\zeta_0^3\\
-3\,\sigma^2 n\zeta_0^3 + 2\,v\zeta_1\zeta_0^2 n\pi_0 + 2\,b\zeta_0^3 + \sigma^2\zeta_0^3 - 2\,a\zeta_1\zeta_0\pi_0\pi_1 + 2\,n^2 b\zeta_0^3 - n^3\sigma^2\zeta_0^3 + 2\,a\zeta_1^2\pi_0^2 = 0.
\end{array}\right\} \quad (15)
$$

When solving system (15) using Maple, the following results are obtained:

$$
\pi_0 = \frac{na\pi_1^2 + c(n-1)^2}{4na\pi_2},\ \zeta_0 = \frac{\pi_1\sqrt{-nac} + c(n-1)}{(n-1)\left[2b + (1-n)\sigma^2\right]},\ \zeta_1 = \frac{2\pi_2\sqrt{-nac}}{(n-1)\left[(1-n)\,\sigma^2 + 2b\right]}, \quad (16)
$$

and

$$
v = \frac{a\left[(1-n)\,\sigma^2 + 2b\right]}{2\sqrt{-nac}}. \quad (17)
$$

The solutions for Eq (1) are as follows:

Case 1. If $\pi_1^2 - 4\pi_0\pi_2 > 0$, then substituting (11) and (16) into (14) yields the combo soliton solution of Eq (1) as:

$$
q(x,t) = \left\{\frac{c}{2b + (1-n)\sigma^2}\left[1 + \frac{A_1\tanh\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x - vt)\right) + A_2}{A_1 + A_2\tanh\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x - vt)\right)}\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right]. \quad (18)
$$

From solution (18), if $A_1 \neq 0$ and $A_2 = 0$, then the dark soliton solution is obtained as follows:

$$
q(x,t) = \left\{\frac{c}{2b + (1-n)\sigma^2}\left[1 + \tanh\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}\,(x - vt)\right)\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \quad (19)
$$

while, if $A_1 = 0$ and $A_2 \neq 0$, then the singular soliton solution is obtained as follows:

$$
q(x,t) = \left\{\frac{c}{2b + (1-n)\sigma^2}\left[1 + \coth\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x - vt)\right)\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right]. \quad (20)
$$

Case 2. If $\pi_1^2 - 4\pi_0\pi_2 < 0$, then substituting (11) and (16) into (14) yields the periodic solution of Eq (1) as:

$$q(x,t)=\left\{\frac{c}{2b+(1-n)\sigma^2}\left[1+\frac{iA_3\tan\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x-vt)\right)-A_4}{A_3-iA_4\tan\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x-vt)\right)}\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right]. \quad (21)$$

From solution (21), if $A_3 \neq 0$ and $A_4 = 0$, then one gets the periodic solution:

$$q(x,t)=\left\{\frac{c}{2b+(1-n)\sigma^2}\left[1+i\tan\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x-vt)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right], \quad (22)$$

while, if $A_3 = 0$ and $A_4 \neq 0$, then one gets the periodic solution:

$$q(x,t)=\left\{\frac{c}{2b+(1-n)\sigma^2}\left[1-i\cot\left(\frac{n-1}{2}\sqrt{-\frac{c}{na}}(x-vt)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right]. \quad (23)$$

In solutions (18)–(23), the velocity v is given by (17).

Case 3. By substituting $\pi_0 = \frac{\pi_1^2}{4\pi_2}$ into the system given by Eq (15) and solving it using Maple, the following results are obtained:

$$\zeta_0 = \frac{\zeta_1\pi_1}{2\pi_2}, \quad (24)$$

and

$$b=(n-1)\sigma^2,\ c=0,\ v=\frac{a\pi_2}{\zeta_1(n-1)}. \quad (25)$$

Substituting (13) and (24) into (14) yields the rational solution of Eq (1) as:

$$q(x,t)=\left(-\frac{\zeta_1}{\pi_2(x-vt)+A_5}\right)^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right]. \quad (26)$$

Solution (26) is satisfied under the constraint conditions (25).

4. The modified Kudryashov method (MKM)

The MKM provides a wide range of solutions, including combo bright-singular solitons, dark solitons, and singular solitons. It is flexible and adaptable to various nonlinear PDEs. The method often offers higher precision and requires fewer assumptions compared to other approaches. Additionally, it effectively deduces singular soliton solutions, which are vital for certain physical models. The MKM cannot deduce bright soliton solutions, limiting its use in some applications. It may require intensive symbolic computations due to its reliance on higher-degree polynomial expansions. Moreover, unlike the UREEM, it may not produce periodic solutions, which are essential in some scenarios, such as wave propagation in periodic media. According to the method described in [32], we assume that Eq (8) has the following formal solution:

$$g(z) = \sum_{l=0}^{N} E_l F^l(z), \tag{27}$$

where E_l are constants, such that $E_N \neq 0$, and $F(z)$ satisfies the auxiliary ODE:

$$F'(z) = F(z)\left[F^r(z) - 1\right]\ln A, \qquad 0 < A \neq 1. \tag{28}$$

We establish the relationship between N and r as follows:

$$D\left[g(z)\right] = N,\ D\left[g'(z)\right] = N + r, D\left[g''(z)\right] = N + 2r. \tag{29}$$

Consequently, in general, we have

$$D\left[g^p(z)\, g^{(s)}(z)\right] = N(p+1) + sr. \tag{30}$$

It is well-known that Eq (28) has the following solution:

$$F(z) = \left[\frac{1}{1 + \varepsilon \exp_A(rz)}\right]^{\frac{1}{r}}, \tag{31}$$

where $\varepsilon = \pm 1$, and r is a positive integer. The solution (31) reduces to the combo bright-singular soliton solution:

$$F(z) = \left[1 - \frac{1}{1 + \sinh\left[rz\ln(A)\right] - \cosh\left[(rz)\ln(A)\right]}\right]^{\frac{1}{r}}, \tag{32}$$

the dark soliton solution:

$$F(z) = \left[\frac{1}{2}\left[1 - \tanh\left(\frac{rz}{2}\ln(A)\right)\right]\right]^{\frac{1}{r}}, \tag{33}$$

and the singular soliton solution:

$$F(z) = \left[\frac{1}{2}\left[1 - \coth\left(\frac{rz}{2}\ln(A)\right)\right]\right]^{\frac{1}{r}}. \tag{34}$$

Balancing $g(z)g''(z)$ with $g^2(z)g'(z)$ in Eq (8), one gets

$$N + N + 2r = 2N + N + r \implies N = r. \tag{35}$$

Case 1. Selecting $r = 1$, we deduce that $N = 1$. It follows that Eq (8) has the solution form as:

$$g(z) = E_0 + E_1 F(z), \tag{36}$$

where E_0 and E_1 are constants, provided $E_1 \neq 0$. Substituting (28) and (36) into Eq (8), one gets the following algebraic equations:

$$
\left.\begin{array}{l}
-2\,vE_1^3\ln(A)+2n\,vE_1^3\ln(A)-2\,aE_1^2\ln^2(A)=0,\\[2ex]
2\,bE_1^3+2\,vE_1^3\ln(A)+2\,aE_1^2\ln^2(A)\,n+3\,n^2\sigma^2E_1^3-n^3\sigma^2E_1^3-4\,aE_1\ln^2(A)\,E_0\\
+\sigma^2E_1^3-3\,\sigma^2nE_1^3+4\,aE_1\ln^2(A)\,E_0n-4\,vE_1^2\ln(A)\,E_0+2\,aE_1^2\ln^2(A)-2\,vE_1^3\ln(A)\,n\\
+4\,vE_1^2\ln(A)\,E_0n+2\,n^2bE_1^3-4\,nbE_1^3=0,\\[2ex]
6\,n^2bE_0E_1^2-2\,vE_1\ln(A)\,E_0^2-12\,nbE_0E_1^2-9\,\sigma^2nE_0E_1^2+2\,vE_1\ln(A)\,E_0^2n\\
-4\,vE_1^2\ln(A)\,E_0n-6\,aE_1\ln^2(A)\,E_0n-2\,cE_1^2+9\,n^2\sigma^2E_0E_1^2-2\,cn^2E_1^2-2\,aE_1^2\ln^2(A)\,n\\
+6\,bE_0E_1^2+3\,\sigma^2E_0E_1^2+4\,cE_1^2n+6\,aE_1\ln^2(A)\,E_0-3\,n^3\sigma^2E_0E_1^2+4\,vE_1^2\ln(A)\,E_0=0,\\[2ex]
6\,bE_0^2E_1+2\,aE_1\ln^2(A)\,E_0n+6\,n^2bE_0^2E_1+2\,vE_1\ln(A)\,E_0^2-4\,cn^2E_0E_1+8\,cE_0E_1n\\
-12\,nbE_0^2E_1-2\,vE_1\ln(A)\,E_0^2n-9\,\sigma^2nE_0^2E_1+3\,\sigma^2E_0^2E_1-3\,n^3\sigma^2E_0^2E_1-4\,cE_0E_1\\
-2\,aE_1\ln^2(A)\,E_0+9\,n^2\sigma^2E_0^2E_1=0,\\[2ex]
-n^3\sigma^2E_0^3+4\,cE_0^2n-2\,cn^2E_0^2-2\,cE_0^2+\sigma^2E_0^3+2\,bE_0^3\\
-4\,nbE_0^3+3\,n^2\sigma^2E_0^3+2\,n^2bE_0^3-3\,\sigma^2nE_0^3=0.
\end{array}\right\} \quad (37)
$$

As a results, one uses Maple to solve system (37) to get

$$E_0=\frac{2c}{2b+(1-n)\sigma^2},\ E_1=-\frac{2c}{2b+(1-n)\sigma^2}, \quad (38)$$

and

$$a=-\frac{c\,(n-1)^2}{n\ln^2(A)},\quad v=\frac{\left[2b-(n-1)\sigma^2\right](n-1)}{2n\ln(A)}. \quad (39)$$

Now, substituting (38) along with (31)–(34) into (36), one gets the solutions of Eq (1) as:

$$q(x,t)=\left\{\frac{2c}{2b+(1-n)\sigma^2}\left[1-\frac{1}{1+\varepsilon\exp_A(x-vt)}\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2t\right], \quad (40)$$

the combo bright-singular soliton solution:

$$q(x,t)=\left\{\frac{2c}{2b+(1-n)\sigma^2}\left[\frac{1}{1+\sinh[(x-vt)\ln(A)]-\cosh[(x-vt)\ln(A)]}\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2t\right], \quad (41)$$

the dark soliton solution:

$$q(x,t)=\left\{\frac{2c}{2b+(1-n)\sigma^2}\left[1+\tanh\left(\frac{1}{2}(x-vt)\ln(A)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2t\right], \quad (42)$$

and the singular soliton solution:

$$q(x,t)=\left\{\frac{2c}{2b+(1-n)\sigma^2}\left[1+\coth\left(\frac{1}{2}(x-vt)\ln(A)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2t\right]. \quad (43)$$

Solutions (40)–(43) are satisfied under the constraint conditions (39).
Case 2. Selecting $r = 2$, we deduce that $N = 2$. It follows that Eq (8) has the solution form as:

$$g(z) = E_0 + E_1 F(z) + E_2 F^2(z), \tag{44}$$

where E_0, E_1, and E_2 are constants, provided $E_2 \neq 0$. Substituting (28) and (44) into Eq (8), one gets algebraic equations, and as a result, we use Maple to solve it to get

$$E_0 = \frac{2c}{2b + (1-n)\sigma^2}, \quad E_1 = 0, \quad E_2 = -\frac{2c}{2b + (1-n)\sigma^2}, \tag{45}$$

and

$$a = -\frac{c\,(n-1)^2}{4n\ln^2(A)}, \quad v = \frac{\left[2b - (n-1)\sigma^2\right](n-1)}{4n\ln(A)}. \tag{46}$$

Now, substituting (45) along with (31)–(34) into (44), one gets the solutions of Eq (1) as:

$$q(x,t) = \left\{\frac{2c}{2b + (1-n)\sigma^2}\left[1 - \frac{1}{1 + \varepsilon \exp_A(2(x - vt))}\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{47}$$

the combo bright-singular soliton solution:

$$q(x,t) = \left\{\frac{2c}{2b + (1-n)\sigma^2}\left[\frac{1}{1 + \sinh[2(x - vt)\ln(A)] - \cosh[2(x - vt)\ln(A)]}\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{48}$$

the dark soliton solution:

$$q(x,t) = \left\{\frac{c}{2b + (1-n)\sigma^2}\left[1 + \tanh((x - vt)\ln(A))\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{49}$$

and the singular soliton solution:

$$q(x,t) = \left\{\frac{c}{2b + (1-n)\sigma^2}\left[1 + \coth((x - vt)\ln(A))\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right]. \tag{50}$$

Solutions (47)–(50) are satisfied under the constraint conditions (46).
General Case. From Cases 1 and 2, we deduce that Eq (1) has the following general solutions:

$$q(x,t) = \left\{\frac{2c}{2b + (1-n)\sigma^2}\left[1 - \frac{1}{1 + \varepsilon \exp_A(r(x - vt))}\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{51}$$

the combo bright-singular soliton solution:

$$q(x,t) = \left\{\frac{2c}{2b + (1-n)\sigma^2}\left[\frac{1}{1 + \sinh[r(x - vt)\ln(A)] - \cosh[r(x - vt)\ln(A)]}\right]\right\}^{\frac{1}{1-n}} \exp\left[\sigma W(t) - \frac{n-1}{2}\sigma^2 t\right], \tag{52}$$

the dark soliton solution:

$$q(x,t)=\left\{\frac{c}{2b+(1-n)\sigma^2}\left[1+\tanh\left(\frac{r}{2}(x-vt)\ln(A)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right], \tag{53}$$

and the singular soliton solution:

$$q(x,t)=\left\{\frac{c}{2b+(1-n)\sigma^2}\left[1+\coth\left(\frac{r}{2}(x-vt)\ln(A)\right)\right]\right\}^{\frac{1}{1-n}}\exp\left[\sigma W(t)-\frac{n-1}{2}\sigma^2 t\right]. \tag{54}$$

The general solutions (51)–(54) are satisfied under the constraint conditions:

$$a=-\frac{c\,(n-1)^2}{r^2 n\ln^2(A)},\quad v=\frac{\left[2b-(n-1)\sigma^2\right](n-1)}{2rn\ln(A)}. \tag{55}$$

Here $r=1,2,3,...,\infty$.

5. Stability analysis

In this section, we will analyze the stability of Eq (1). To find the steady-state solutions, assume $g(z)$ is constant [33], and let

$$g(z)=g_s. \tag{56}$$

At a steady state,

$$\frac{d}{dz}g(z)=0 \quad\text{and}\quad \frac{d^2}{dz^2}g(z)=0, \tag{57}$$

so, Eq (8) simplifies to

$$(n-1)^2\left\{-2c+\left[2b-(n-1)\sigma^2\right]g_s\right\}g_s^2=0. \tag{58}$$

This gives two possible solutions:

$$g_s=0 \quad\text{or}\quad g_s=\frac{2c}{2b-(n-1)\sigma^2},\ \text{provided}\ 2b-(n-1)\sigma^2\neq 0. \tag{59}$$

These solutions represent the potential equilibrium points where the system can remain stable.

To analyze the stability of these steady-state solutions, let us consider a small perturbation around the steady state g_s. Assume that

$$g(z)=g_s+\epsilon\,\tilde{g}(z), \tag{60}$$

where ϵ is a small parameter, and $\tilde{g}(z)$ is the perturbation. Substitute (60) into Eq (8), and one gets

$$\begin{aligned}&2a(n-1)\left[g_s+\epsilon\,\tilde{g}(z)\right]\frac{d^2}{dz^2}\left[g_s+\epsilon\,\tilde{g}(z)\right]+2a(1-2n)\left(\frac{d}{dz}\left[g_s+\epsilon\,\tilde{g}(z)\right]\right)^2\\&+2v(n-1)\left[g_s+\epsilon\,\tilde{g}(z)\right]^2\tfrac{d}{dz}\left[g_s+\epsilon\,\tilde{g}(z)\right]-2c(n-1)^2\left[g_s+\epsilon\,\tilde{g}(z)\right]^2\\&+(n-1)^2\left[2b-(n-1)\sigma^2\right]\left[g_s+\epsilon\,\tilde{g}(z)\right]^3=0.\end{aligned} \tag{61}$$

Here

$$
\left.
\begin{aligned}
&[g_s + \epsilon\,\tilde{g}(z)]\,\frac{d^2}{dz^2}\,[g_s + \epsilon\,\tilde{g}(z)] = \epsilon\,g_s\tfrac{d^2}{dz^2}\tilde{g}(z) + \epsilon^2\tilde{g}(z)\tfrac{d^2}{dz^2}\tilde{g}(z),\\
&\left(\frac{d}{dz}\,[g_s + \epsilon\,\tilde{g}(z)]\right)^2 = \left(\tfrac{d}{dz}\epsilon\tilde{g}(z)\right)^2 = \epsilon^2\left(\tfrac{d}{dz}\tilde{g}(z)\right)^2,\\
&[g_s + \epsilon\,\tilde{g}(z)]^2\,\tfrac{d}{dz}\,[g_s + \epsilon\,\tilde{g}(z)] = \epsilon g_s^2\tfrac{d}{dz}\tilde{g}(z) + 2\epsilon^2 g_s\tilde{g}(z)\tfrac{d}{dz}\tilde{g}(z) + \epsilon^3\tilde{g}^2(z)\tfrac{d}{dz}\tilde{g}(z),\\
&[g_s + \epsilon\,\tilde{g}(z)]^2 = g_s^2 + 2\epsilon g_s\tilde{g}(z) + \epsilon^2\tilde{g}^2(z),\\
&[g_s + \epsilon\,\tilde{g}(z)]^3 = g_s^3 + 3\epsilon g_s^2\tilde{g}(z) + 3\epsilon^2 g_s\tilde{g}^2(z) + \epsilon^3\tilde{g}^3(z).
\end{aligned}
\right\} \tag{62}
$$

Since ϵ is small, we linearize the equation by keeping only the first-order terms in ϵ, and one derive:

$$
2a(n-1)g_s\frac{d^2}{dz^2}\tilde{g}(z)+2v(n-1)g_s^2\frac{d}{dz}\tilde{g}(z)+\left\{-4c(n-1)^2g_s+3(n-1)^2\left[2b-(n-1)\sigma^2\right]g_s^2\right\}\tilde{g}(z)=0. \tag{63}
$$

To determine the stability, we assume

$$
\tilde{g}(z) = e^{\lambda z}, \tag{64}
$$

where λ is a constant. Substituting (64) into the linearized Eq (63) gives us the characteristic equation:

$$
2a(n-1)g_s\,\lambda^2 + 2v(n-1)g_s^2\,\lambda - 4c(n-1)^2g_s + 3(n-1)^2\left[2b-(n-1)\sigma^2\right]g_s^2 = 0. \tag{65}
$$

Next, we will check the two equilibrium points:

Case 1. Substitute $g_s = 0$ into (65) to find the characteristic equation. At $g_s = 0$, the system's characteristic equation does not give meaningful information about the stability, suggesting neutral stability. Without a linear restoring force, small perturbations neither grow nor decay, implying that $g_s = 0$ is a marginally stable equilibrium point. The detailed behavior of the system might depend on nonlinear effects or external factors.

Case 2. Substitute the non-zero steady state $g_s = \frac{2c}{2b-(n-1)\sigma^2}$ into the linearized Eq (65), and one gets the characteristic equation with these coefficients as:

$$
a\lambda^2 + \frac{2cv}{2b-(n-1)\sigma^2}\lambda + c(n-1) = 0. \tag{66}
$$

The roots of the characteristic equation λ_1 and λ_2 are given by

$$
\lambda = \frac{-\frac{2cv}{2b-(n-1)\sigma^2} \pm \sqrt{\frac{4c^2v^2}{\left[2b-(n-1)\sigma^2\right]^2} - 4ac(n-1)}}{2a}. \tag{67}
$$

Thus, the stability of the steady state depends on the real parts of these roots:

(1) For the system to be stable: Both roots must be negative and the following condition should be met:

$$
\frac{4c^2v^2}{[2b-(n-1)\sigma^2]^2} - 4ac(n-1) > 0 \;\Rightarrow\; cv^2 > a(n-1)\left[2b-(n-1)\sigma^2\right]^2. \tag{68}
$$

This inequality provides a condition on the parameters a, b, c, v, n, and σ for the equilibrium point $g_s = \frac{2c}{2b-(n-1)\sigma^2}$ to be stable.

(2) For the system to be unstable: One root is positive or the following condition is satisfied:

$$\frac{4c^2v^2}{\left[2b-(n-1)\sigma^2\right]^2} - 4ac(n-1) < 0 \;\Rightarrow\; cv^2 < a(n-1)\left[2b-(n-1)\sigma^2\right]^2. \tag{69}$$

(3) For the system to be marginally unstable: The following condition is satisfied:

$$\frac{4c^2v^2}{\left[2b-(n-1)\sigma^2\right]^2} - 4ac(n-1) = 0 \;\Rightarrow\; cv^2 = a(n-1)\left[2b-(n-1)\sigma^2\right]^2, \tag{70}$$

leading to repeated real roots.

The analysis shows that the trivial steady-state solution $g_s = 0$ represents a simple equilibrium point, whose stability depends on specific system parameters. In contrast, the non-trivial steady-state solution $g_s = \frac{2c}{2b-(n-1)\sigma^2}$ requires a more nuanced analysis, as the stability is influenced by a combination of factors including the coefficients a, b, c, v, n, and σ. Understanding these dependencies is crucial for predicting the behavior of the system under small perturbations.

6. Results and discussion

Solitons are self-reinforcing solitary waves that preserve their shape and travel at a constant velocity. These solutions to specific nonlinear partial differential equations are observed in various physical systems, including fluid dynamics, plasma physics, and optical fibers. A crucial feature of solitons is their ability to retain their shape even after interacting with other solitons or external forces, making them a central topic in nonlinear wave theory.

In this section, we compare soliton solutions for different values of σ, focusing on how the soliton's behavior changes as external forcing becomes more pronounced. This comparison highlights the soliton's resilience and eventual breakdown as external influences increase. We simulate various figures for different noise strengths σ for the dark soliton solution (Eq (19), with $a = -1$, $b = 1$, $c = 1$, $n = 2, n = 5$, and $W(t) = \cos(t)$, refer to Figures 1–3) and the combo-bright-singular soliton solution (Eq (41), with $a = -1$, $b = 1$, $c = 1$, $n = 2, n = 5$, $A = 3$, and $W(t) = \cos(t)$, see Figures 4–6).

The analysis explores the behavior of specific soliton solutions under the influence of stochastic or external forcing, represented by σ. By comparing solutions for different values of σ, we gain insights into how varying levels of external noise or forcing influence the soliton's dynamics. We consider three cases of $\sigma = 0, 0.1, 0.2$, representing no external influence, moderate external influence, and strong external influence, respectively.

The physical interpretation of these solutions reveals the soliton's stability, amplitude modulation, and changes in velocity under various conditions. As σ increases, the soliton transitions from a stable, undisturbed wave to one that undergoes significant oscillations, eventually decaying due to strong external forces. This analysis deepens our understanding of soliton behavior in environments with varying levels of stochasticity, ranging from ideal undisturbed settings to highly turbulent or noisy media.

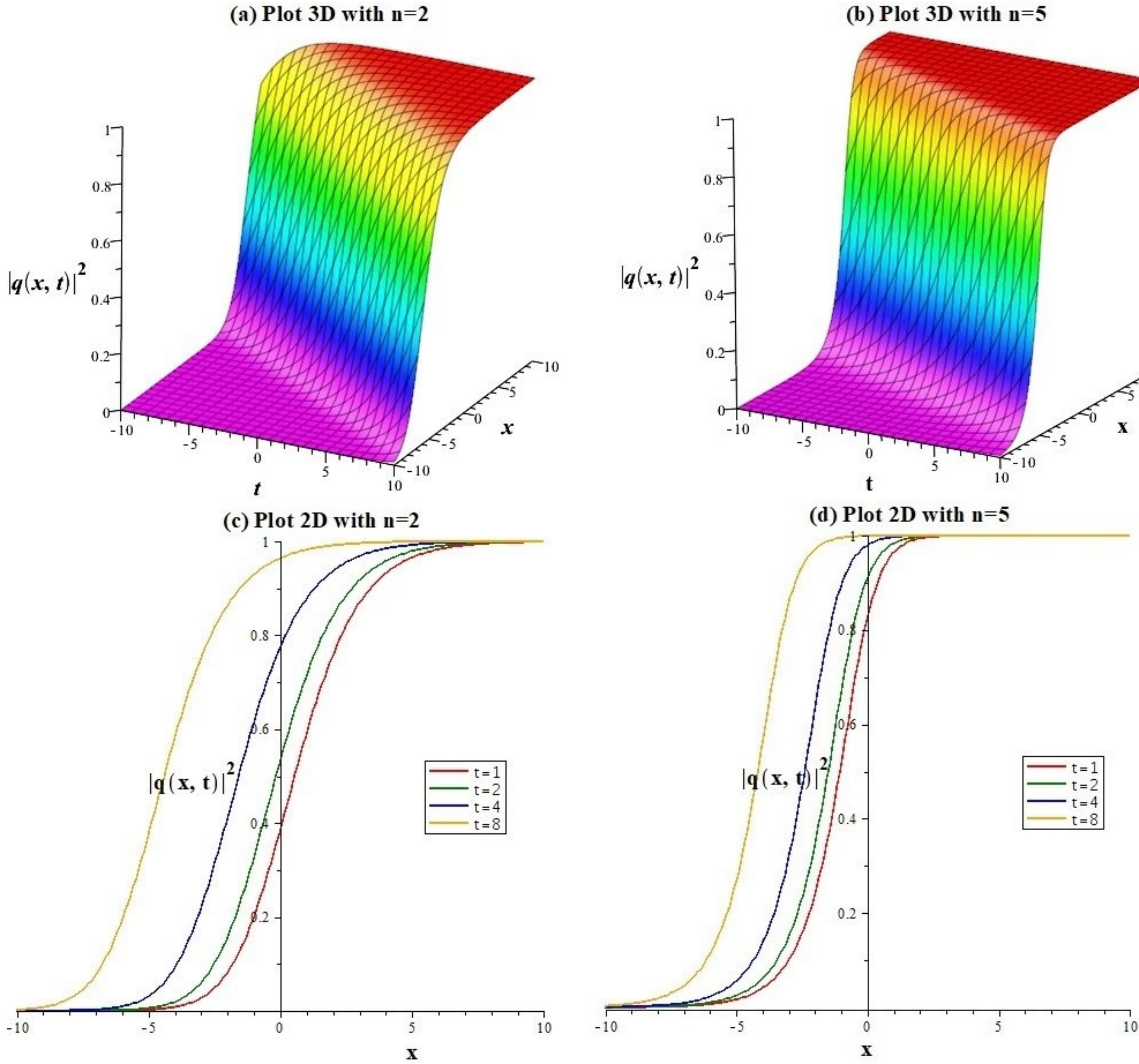

Figure 1. Plots of the dark soliton solution (19), for $\sigma = 0$. The stochastic term $W(t)$ has no effect, resulting in a stable, undisturbed soliton. The soliton maintains its amplitude and shape, propagating smoothly over time. This represents a classical soliton moving through a medium with constant parameters, unaffected by external forces or randomness.

Applications:

- Optical fiber communications: Solitons propagate without disturbance.
- Shallow water waves: Solitary waves in undisturbed conditions.
- Stable energy or information transmission over long distances.

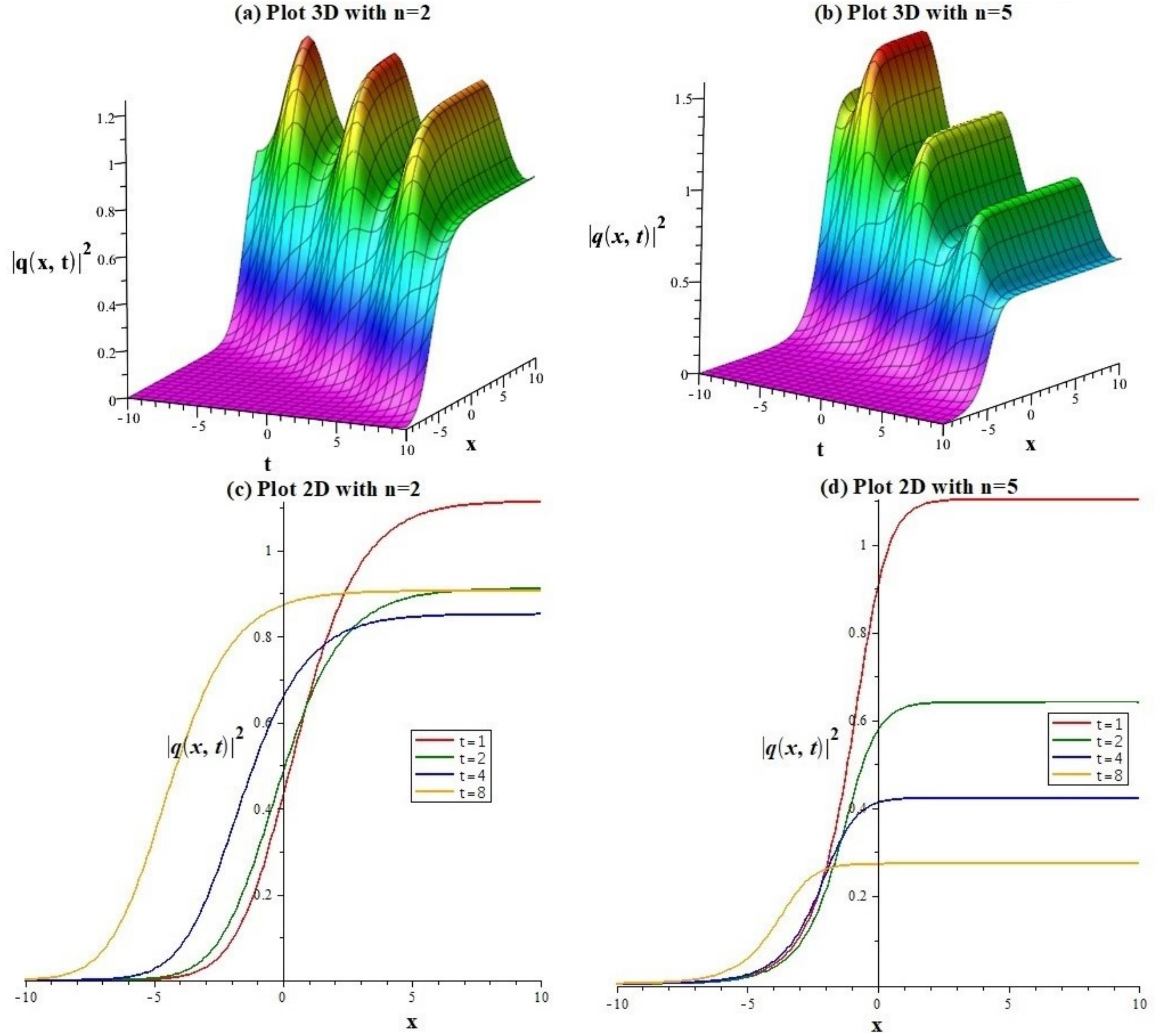

Figure 2. Plots of the dark soliton solution (19), for $\sigma = 0.1$. The soliton's profile becomes more intricate. Its amplitude exhibits oscillations, leading to periodic stretching and compression. This behavior models a soliton traveling through a medium influenced by external oscillations or random forces.

Applications:

- Modeling wave propagation in optical fibers under moderate noise.
- Describing nonlinear waves in plasmas exposed to external electric fields.
- Quantum systems with particles interacting with moderate noise.

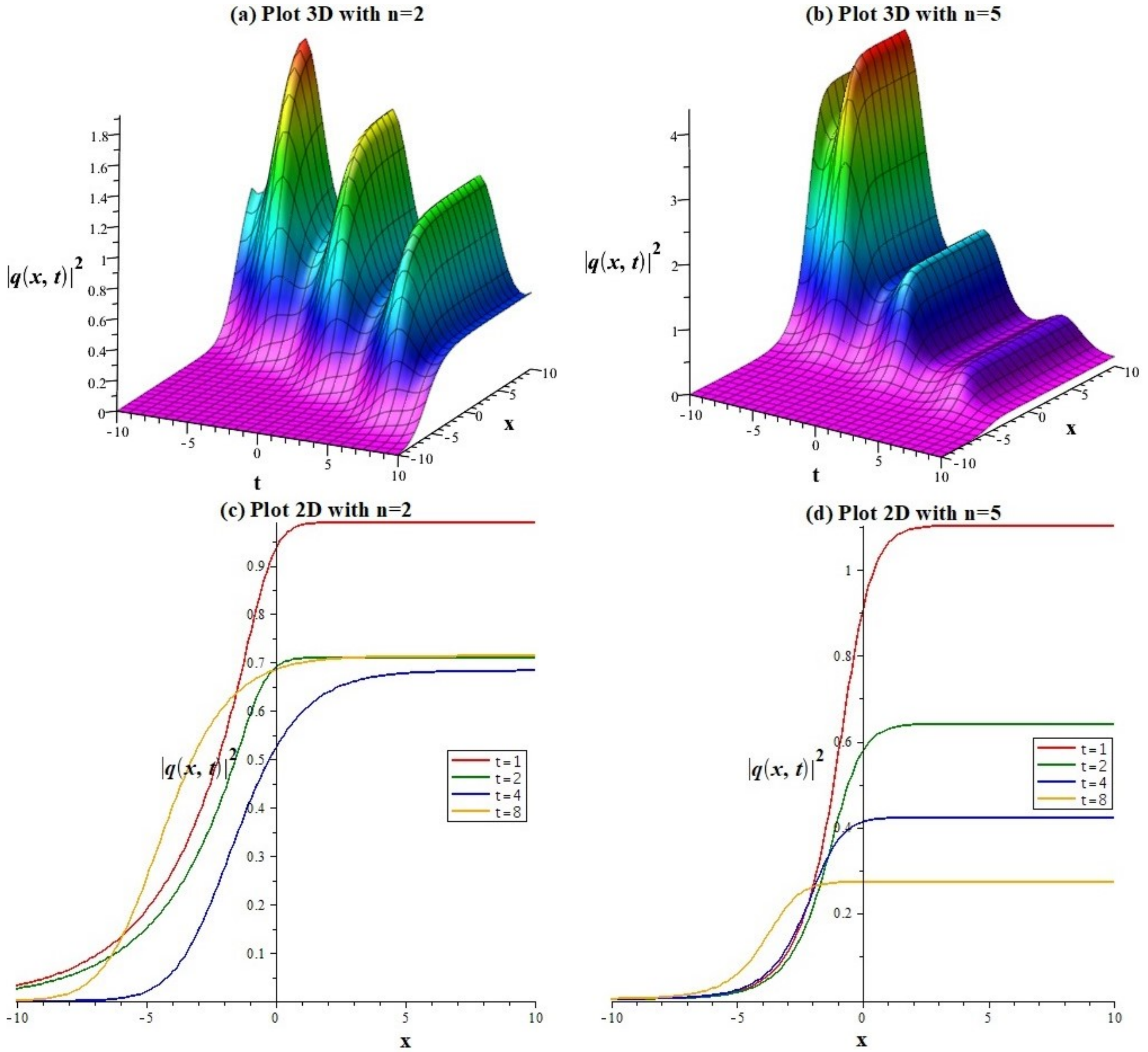

Figure 3. Plots of the dark soliton solution (19), for $\sigma = 0.2$. The soliton undergoes large oscillations and rapid dissipation, with its shape and energy quickly deteriorating. This behavior reflects a soliton in a highly damped or noisy environment.

Applications:

- Fluid dynamics modeling: Turbulence under strong stochastic forces.
- Signal degradation: Optical systems exposed to heavy noise.
- Describing perturbed plasma waves or Bose-Einstein condensates under significant external influence.

Dark soliton solutions are critical for understanding nonlinear wave phenomena across many scientific and technological fields. Their unique properties, such as localized amplitude dips and stable propagation, make them valuable for applications in fiber optics, Bose-Einstein condensates, fluid dynamics, plasma physics, and more. These solitons provide insights into the fundamental behavior of nonlinear systems while offering practical solutions to real-world challenges like improving data transmission, controlling light in photonic devices, and modeling wave dynamics in complex media. As research advances, the applications of dark solitons are likely to expand, driving innovations in quantum computing, metamaterials, and nonlinear acoustic systems.

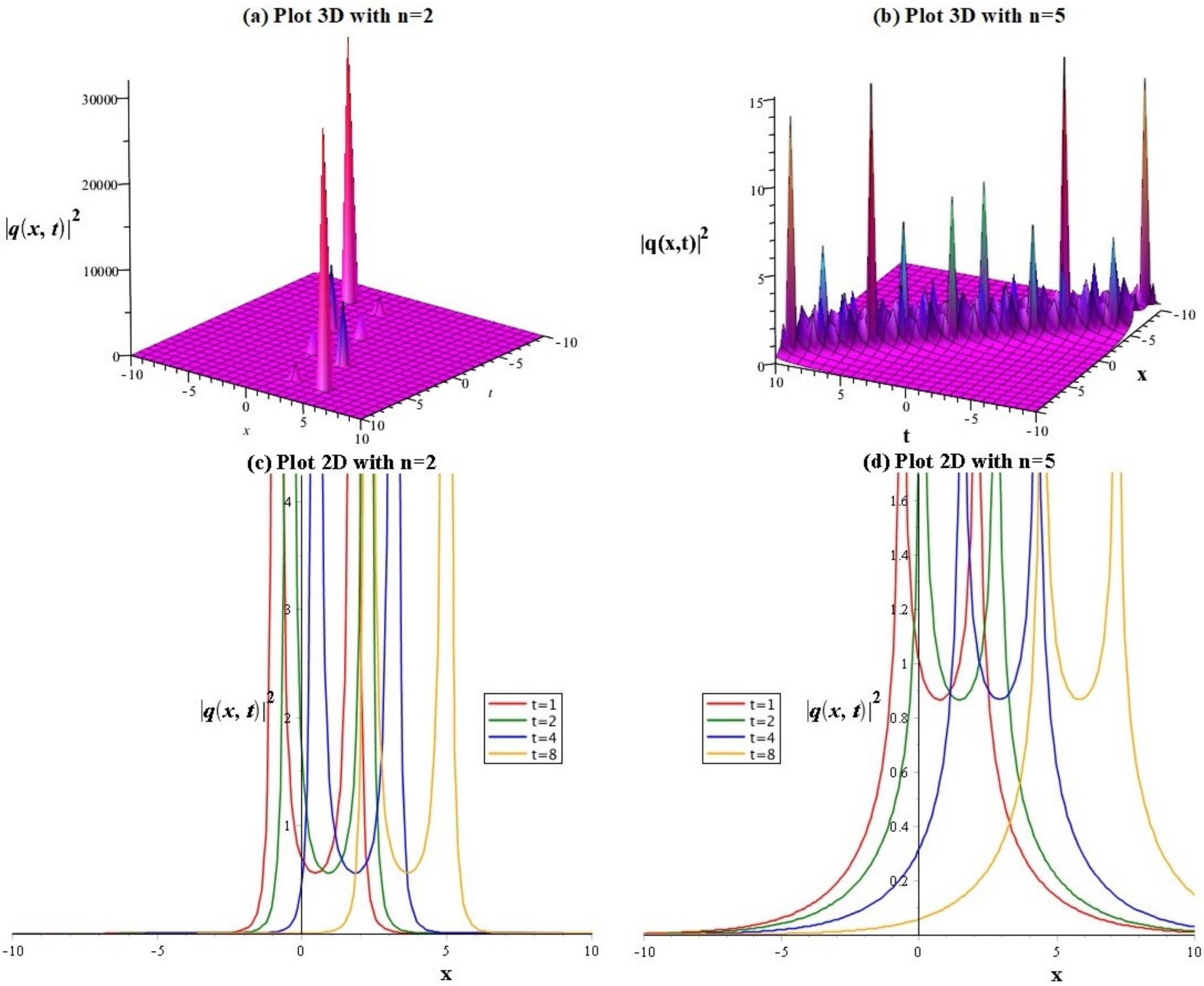

Figure 4. Plots of the combo-bright-singular soliton solution (41), for $\sigma = 0$. The soliton remains unaffected by the stochastic term $W(t)$. The solution is symmetric, with the soliton maintaining a stable shape and amplitude during propagation. This represents a soliton moving through an ideal, undisturbed medium, traveling steadily without modulation or damping.

Applications:

- Fiber optics: Transmission of undisturbed optical solitons.
- Shallow water waves: Solitary water waves in stable environments.
- Mechanical systems: Stability in soliton transport in coupled oscillators.

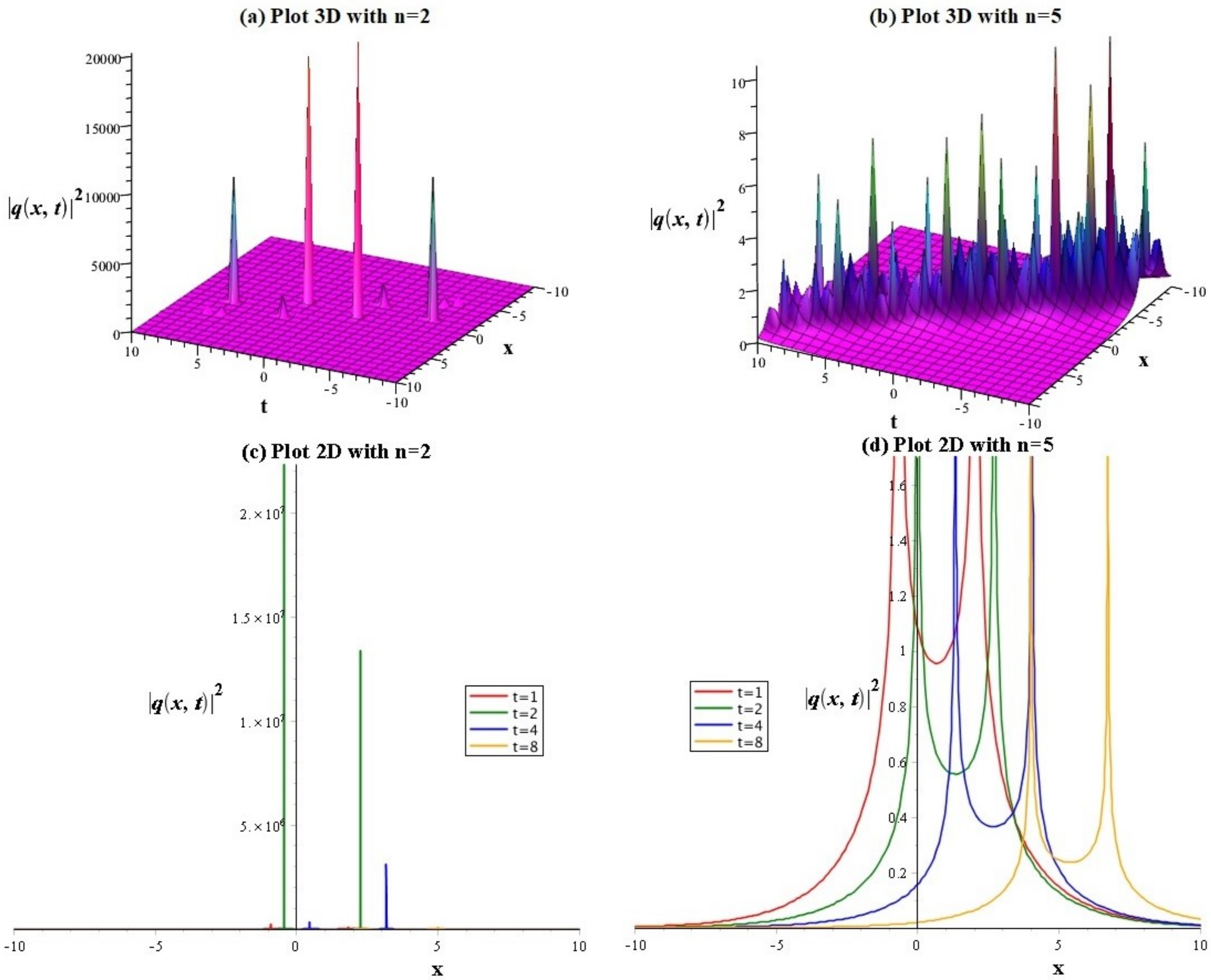

Figure 5. Plots of the combo-bright-singular soliton solution (41), for $\sigma = 0.1$. The soliton retains its general wave shape but exhibits slight oscillations, with periodic stretching or compression. This represents a soliton influenced by mild external forces or fluctuations, maintaining relative stability.

Applications:

- Plasma physics: Modeling wave dynamics in moderately noisy plasma environments.
- Nonlinear acoustics: Propagation of sound waves in materials with minor imperfections.
- Quantum systems: Describing quantum solitons under weak external noise.

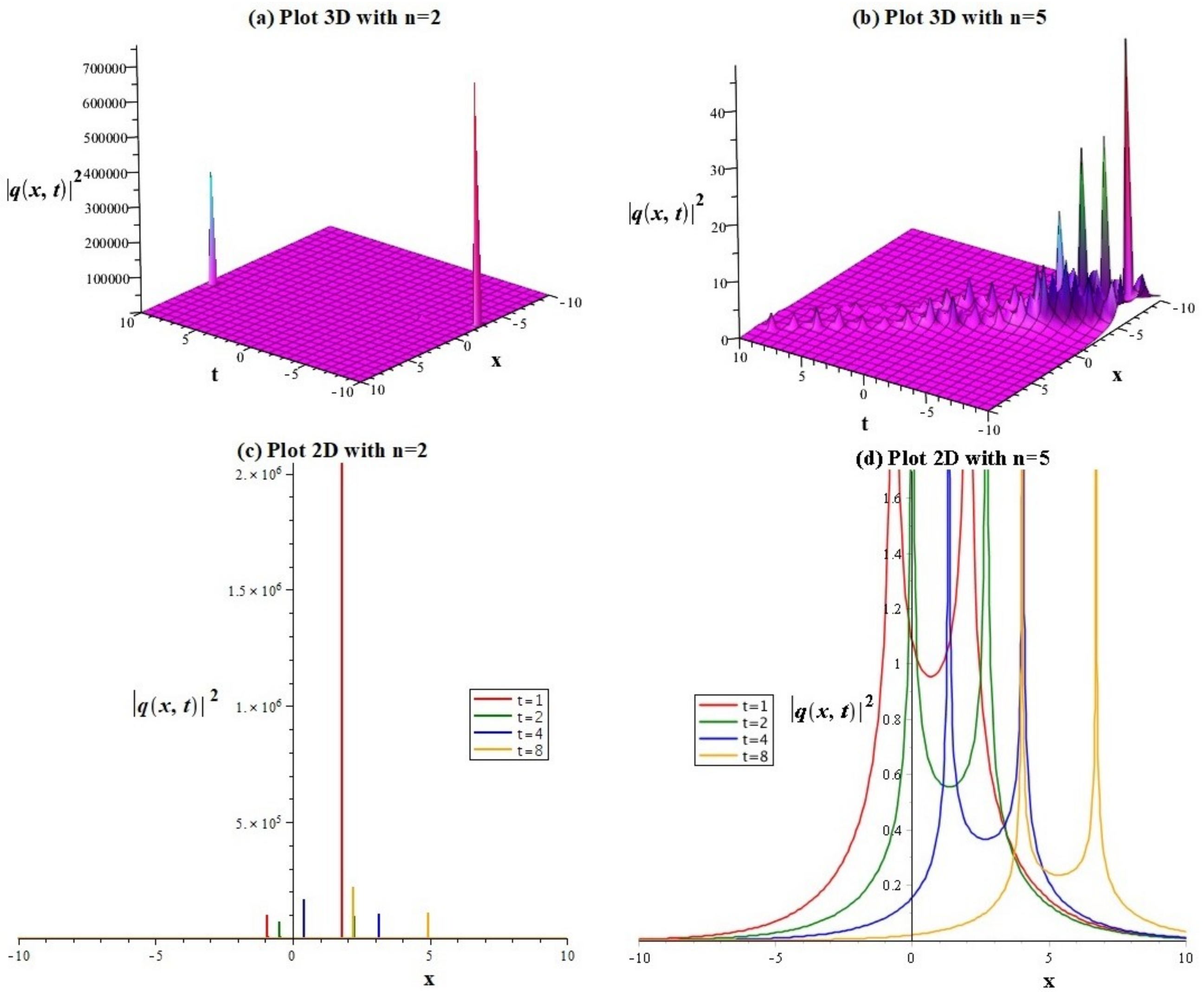

Figure 6. Plots of the combo-bright-singular soliton solution (41), for $\sigma = 0.2$. The soliton's shape becomes distorted, exhibiting large oscillations and rapid energy dissipation. This reflects a soliton under strong external influence, where large oscillations and high noise levels dominate, causing the soliton to break down.

Applications:

- Fluid dynamics: Modeling turbulence in high-noise environments.
- Signal processing: Analyzing noisy signals in communication systems.
- Wave propagation in random media: Modeling seismic or oceanic waves in irregular environments.

For $\sigma = 0$: The intensity is lower, and the wave propagation is more localized.
For $\sigma = 0.1$: The intensity increases moderately, with the spatial and temporal structure becoming more pronounced.
For $\sigma = 0.2$: The intensity increases significantly, leading to wider regions of high intensity, as the σ-dependent term $\exp[\sigma W(t) - ((n-1)/2)\sigma^2 t]$ becomes more significant.

This plot illustrates the wave-like nature of the solution, showing how the intensity evolves dynamically as a function of time and space, with higher σ values leading to greater intensity and larger spatial and temporal spread.

7. Conclusions

In this study, we conducted an in-depth investigation into the behavior of solitons within the stochastic nonlinear reaction-diffusion equation (RDE) with multiplicative white noise. Utilizing the UREEM and MKM methods, we derived a variety of soliton solutions, including combo-dark solitons, dark solitons, singular solitons, combo-bright-singular solitons, and periodic wave solutions. These solutions provide crucial insights into the dynamics of solitons in stochastic nonlinear systems.

We analyzed the soliton solutions for different values of σ to explore the impact of external or stochastic forces. For $\sigma = 0$, the soliton remained stable, representing a classical soliton traveling through an ideal, undisturbed medium, with constant shape and amplitude and no external forces affecting its propagation. At $\sigma = 0.1$, moderate external influences resulted in periodic oscillations in the soliton's amplitude and position, reflecting mild modulation without significant distortion. This behavior suggests a soliton moving through a medium influenced by weak external or stochastic forces. For $\sigma = 0.2$, the soliton experienced substantial external forces, leading to large oscillations, rapid decay in amplitude, and significant distortion of its position. This indicated the breakdown of the soliton's stability under high external influences, causing the soliton to lose coherence and dissipate quickly.

These findings highlight the progressive destabilization of solitons as the strength of external influences, represented by σ, increases. Starting from stable, coherent behavior, the soliton transitions to rapid dissipation and distortion under stronger external forces. Furthermore, our stability analysis revealed that certain soliton solutions preserve their structural integrity even in the presence of multiplicative noise, demonstrating their resilience in stochastic environments.

This study makes a significant contribution to understanding soliton dynamics in stochastic systems, with potential applications in various scientific and engineering disciplines. Future research could explore more complex types of noise and extend these methods to other classes of stochastic partial differential equations, further advancing the understanding of soliton behavior in diverse environments.

Author contributions

Nafissa T. Trouba: Formal analysis, Data curation, Funding acquisition; Huiying Xu: Supervision, Investigation, Visualization; Mohamed E. M. Alngar: Conceptualization, Methodology, Writing-review & editing, Writing-original draft, Resources, Software; Reham Shohib: Formal analysis, Data curation; Haitham A. Mahmoud: Funding acquisition, Validation, Project administration; Xinzhong Zhu: Formal analysis, Conceptualization, Investigation, Visualization. All the authors have agreed and given their consent for the publication of this research paper.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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